Section 3.7
The Chain Rule

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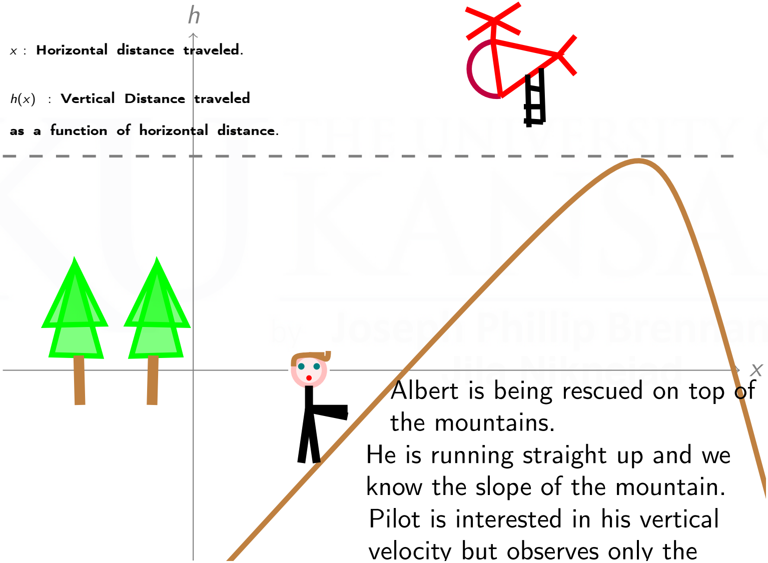
(1) Compositions
(2) The Chain Rule

Example I

x : **Horizontal distance traveled.**

$h(x)$: **Vertical Distance traveled as a function of horizontal distance.**



Albert is being rescued on top of the mountains.
He is running straight up and we know the slope of the mountain.
Pilot is interested in his vertical velocity but observes only the

Review: Composition of Functions

If f and g are functions, then their **compositions** $f \circ g$ and $g \circ f$ are:

$$(f \circ g)(x) = f(g(x)) \qquad\qquad (g \circ f)(x) = g(f(x))$$

For example, if $f(u) = \sqrt{u}$ and $g(x) = \dfrac{1}{x-3}$ then

$$(f \circ g)(x) = \sqrt{\frac{1}{x-3}} \qquad\qquad (g \circ f)(u) = \frac{1}{\sqrt{u}-3}$$

Note: Composition should be read "inside out" — when you calculate $(f \circ g)(x) = f(g(x))$, you apply g (the innermost function) first.

It's possible to compose more than two functions at a time:

$$(f \circ g \circ h)(x) = f(g(h(x))) \qquad (f \circ g \circ h \circ k)(x) = f(g(h(k(x))))$$

The Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $H = f \circ g$ defined by $H(x) = f(g(x))$ is differentiable at x, and its derivative is

$$H'(x) = f'(g(x))\, g'(x) \ .$$

$$\underbrace{f'}_{\text{Outside Derivative}} \quad \underbrace{(g(x))}_{\text{(Inside Untouched)}} \quad \times \quad \underbrace{g'(x)}_{\text{Inside Derivative}}$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx} \ .$$

Example II

(I) To find $\frac{d}{dx}\left(\sqrt{x^3+x}\right)$:

Step 1: Write $\sqrt{x^3+x}$ as a composite function $f(g(x))$.

$$y = f(u) = \sqrt{u} \qquad \frac{dy}{du} = f'(u) = \frac{1}{2\sqrt{u}}$$

$$u = g(x) = x^3 + x \qquad \frac{du}{dx} = g'(x) = 3x^2 + 1$$

Step 2: Apply the Chain Rule.

$$\frac{d}{dx}\left(\sqrt{x^3+x}\right) = f'(g(x))\,g'(x) = \frac{dy}{du}\frac{du}{dx}$$

$$= \left(\frac{1}{2\sqrt{x^3+x}}\right)(3x^2+1) = \boxed{\frac{3x^2+1}{2\sqrt{x^3+x}}}$$

Example II:

(I) $$\frac{d}{dx}\left(\sqrt{x^3+x}\right) = \frac{3x^2+1}{2\sqrt{x^3+x}}$$

(II) $$\frac{d}{dx}\left((2x+1)^3(x^2-1)^{-2} \right) = \frac{-2(2x+1)^2(x^2+2x+3)}{(x^2-1)^3}$$

(III) $$\frac{d}{dx}\left(e^{\frac{x}{x^2+1}}\right) = e^{\frac{x}{x^2+1}}\left(\frac{-(x-1)(x+1)}{(x^2+1)^2}\right)$$

(IV) $$\frac{d}{dz}\left(9^{1-z^4}\right) = 9^{1-z^4}\ln(9)(-4z^3)$$

Example III, Applications of the Chain Rule

As air is pumped into a balloon, the volume and the radius increase. Both volume (V) and radius (r) are functions of time. The volume formula for a sphere also expresses the volume as a function of radius:

$$V(r) = \frac{4\pi}{3} r^3$$

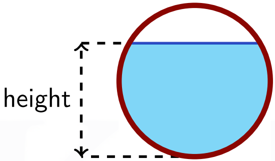
Since radius is a function $r(t)$ of time t, the dependence of volume on time can be expressed as a composite function:

$$V(r(t)) = \frac{4\pi}{3} \big(r(t)\big)^3$$

At $t = 1$ minute the radius is 3 inches and growing at a rate of 2 inches per minute. How fast is the volume of the balloon growing?

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi\big(r(t)\big)^2 r'(t) = 4\pi(3)^2(2) = 72\pi \text{ in}^3/\text{min}$$

Example IV, Applications of the Chain Rule



If a spherical tank of radius 4 feet is filled to a height of h feet, then the volume V of water in the tank (in feet^3) is given by the formula

$$V(h) = \pi\left(4h^2 - \frac{h^3}{3}\right) \implies V'(h) = \pi(8h - h^2)$$

The tank is regulated by a faucet and a drain so that the height h of water at time t (in hours) is given by

$$h(t) = t^2 + t \qquad \implies \qquad h'(t) = 2t + 1$$

At $t = 2$ hours the height of water is $h(2) = 6$ ft and is changing at $h'(2) = 5$ ft/hr. The volume of water is changing at $V'(6) = 12\pi$ ft^3 per foot of water height. Therefore, the rate of change of volume is

$$\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt} = \left(12\pi\frac{\text{ft}^3}{\text{ft}}\right)\left(5\frac{\text{ft}}{\text{hour}}\right) = 60\pi\frac{\text{feet}^3}{\text{hour}}$$

The Power Rule and the Chain Rule

If n is any real number and $u = g(x)$ is differentiable, then

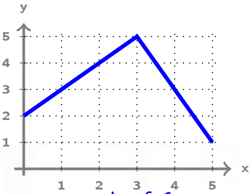
$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx} \qquad \frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}g'(x)$$

Example V

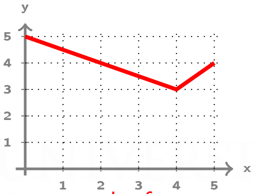
(I) $\frac{d}{dx}\left((x^3-1)^{100}\right) = 300x^2(x^3-1)^{99}$

(II) $\frac{d}{dx}\left(\sqrt{f(x)}\right) = \frac{f'(x)}{2\sqrt{f(x)}}$

Example VI



graph of f



graph of g

$$f'(x) = \begin{cases} 1 & \text{if } 0 < x < 3, \\ -2 & \text{if } 3 < x < 5. \end{cases} \qquad g'(x) = \begin{cases} -1/2 & \text{if } 0 < x < 4, \\ 1 & \text{if } 4 < x < 5. \end{cases}$$

(I) Let $h(x) = f(g(x))$. Find $h'(2)$.

$$h'(2) = f'(g(2))g'(2) = f'(4)g'(2) = (-2)(-1/2) = \boxed{1}$$

(II) Let $h(x) = f(x^2)g(x+1)$. Find $h'(2)$.

$$h'(x) = f(x^2)g'(x+1) + 2xf'(x^2)g(x+1)$$

$$h'(2) = \underbrace{f(4)}_{3}\,\underbrace{g'(3)}_{-1/2} + 4\,\underbrace{f'(4)}_{-2}\,\underbrace{g(3)}_{3.5} \implies \boxed{h'(2) = -29.5}$$